

# Quantum phase transition in a transverse Ising chain with regularly varying parameters \*

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Using rigorous analytical analysis and exact numerical data for the spin- $\frac{1}{2}$  transverse Ising chain we discuss the effects of regular alternation of the Hamiltonian parameters on the quantum phase transition inherent in the model.

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The one-dimensional spin- $\frac{1}{2}$  Ising model in a transverse field (the transverse Ising chain) defined by the Hamiltonian

$$H = \sum_n 2I s_n^x s_n^x + \sum_n \Omega s_n^z \quad (1)$$

is known to be the simplest system exhibiting a quantum (zero-temperature) phase transition driven by the transverse field [1]. Most of the performed studies for this model use the exact eigenvalues and eigenfunctions of its Hamiltonian (1) that makes the problem amenable for rigorous analysis [1, 2]. It is generally known that the critical value of the transverse field is  $\Omega_c = |I|$  (and  $\Omega_c = -|I|$ ). The longitudinal (Ising) magnetization per site  $m^x = \frac{1}{N} \sum_n \langle s_n^x \rangle$  is the order parameter of the system.  $|m^x|$  varies from  $\frac{1}{2}$  (for  $\Omega = 0$ ) to 0 (for  $\Omega \geq \Omega_c$ ) according to  $|m^x| = \frac{1}{2} \left( 1 - \left( \frac{\Omega}{\Omega_c} \right)^2 \right)^{\frac{1}{8}}$ . The quantum phase transition at  $\Omega_c$  is equivalent to the thermal phase transition of the square-lattice Ising model. After the understanding of the properties of the basic model was achieved the models with various modifications were introduced and the effects of introduced changes on the quantum phase transition were discussed. Among numerous works in this field one may mention

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an analysis of the critical behaviour of the aperiodic transverse Ising chain by J.M.Luck [3], an extensive real-space renormalization-group treatment of the random transverse Ising chain by D.S.Fisher [4] or renormalization-group study of the aperiodic transverse Ising chain by F.Iglói et al [5]. It should be remarked, however, that the study of a simpler case of the *regularly inhomogeneous* transverse Ising chain is still lacking although many properties of such a system can be examined exploiting fermionic representation either rigorously analytically (using continued fractions) or on a very precise and well-controlled level of approximation numerically (studying long chains).

In this brief report (for the extended version see [6]) we highlight how deviations from the pure uniform crystalline system in kind of regularly alternating exchange interactions and transverse fields influence the quantum phase transition inherent in the spin- $\frac{1}{2}$  transverse Ising chain. The regular alternation is obtained by substituting in (1)  $I_n$  and  $\Omega_n$  instead of  $I$  and  $\Omega$  assuming a periodic sequence

$$I_1\Omega_1 \dots I_p\Omega_p I_1\Omega_1 \dots I_p\Omega_p \dots$$

In particular, we present the results of rigorous analytical study based on [7] for thermodynamic quantities and of exact numerical study using the method illustrated in [8] for spin correlations. Our main conclusions are that the number of second-order quantum phase transitions for a given period of alternation  $p$  strongly depends on the concrete values of the Hamiltonian parameters whereas the critical behaviour remains as for the uniform chain. Moreover, for a certain values of the Hamiltonian parameters weaker singularities of the ground-state quantities may appear.

We start from recalling an old result of P.Pfeuty [9] in the present context. P.Pfeuty showed that for the spin- $\frac{1}{2}$  transverse Ising chain Hamiltonian the gap in the excitation spectrum (in the thermodynamic limit) is zero at the “critical point”

$$\prod_n I_n = \prod_n \Omega_n. \quad (2)$$

A result similar to (2) can be also derived by the continued fraction approach developed in Ref. [7]. In particular, for a chain of period 2 (period 3) Eq. (2) yields  $I_1 I_2 = \pm \Omega_1 \Omega_2$  ( $I_1 I_2 I_3 = \pm \Omega_1 \Omega_2 \Omega_3$ ). Consider further chains with regularly alternating transverse field  $\Omega_n = \Omega + \Delta \Omega_n$ ,  $\Delta \Omega_1 + \dots + \Delta \Omega_p = 0$ ,  $|I_n| = |I|$  ( $= 1$ ). For a chain of period 2 ( $\Delta \Omega_1 = -\Delta \Omega_2 = \Delta \Omega > 0$ ) the equation  $\Omega_c^2 - \Delta \Omega^2 = \pm I^2$  yields either two critical transverse fields,  $\pm \sqrt{\Delta \Omega^2 + I^2}$  (if  $\Delta \Omega < |I|$ ), or three critical transverse fields,  $-\sqrt{2}\Delta \Omega$ ,  $0$ ,  $\sqrt{2}\Delta \Omega$  (if  $\Delta \Omega = |I|$ ), or four critical transverse fields,  $\pm \sqrt{\Delta \Omega^2 + I^2}$

and  $\pm\sqrt{\Delta\Omega^2 - I^2}$  (if  $\Delta\Omega > |I|$ ). Thus, the number of critical transverse fields yielding gapless energy spectrum of the Ising chain in a modulated transverse field of period 2 depends on a strength of inhomogeneity. For small  $\Delta\Omega$  only quantitative changes with respect to the homogeneous case may be expected, whereas for large  $\Delta\Omega$  some qualitative changes should occur. In Fig. 1 we report a behaviour of the ground-state longitudinal

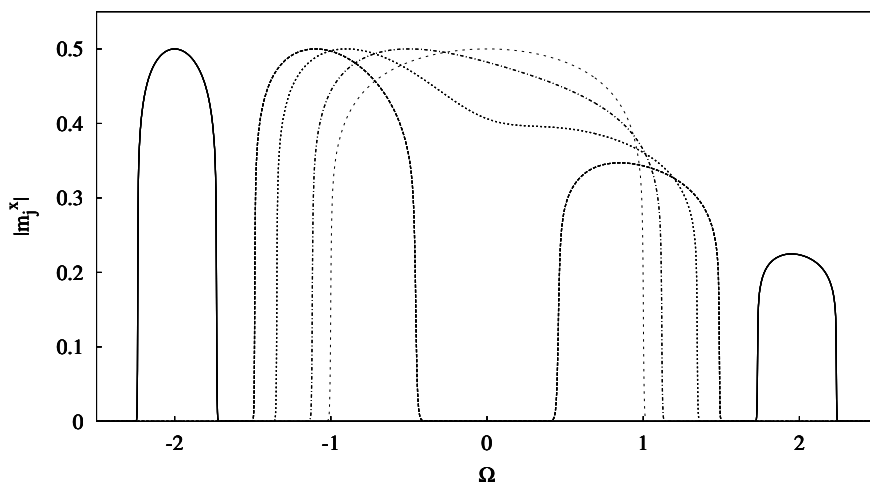


Fig. 1. The ground-state longitudinal sublattice magnetization of the spin- $\frac{1}{2}$  transverse Ising chain with  $|I_1| = |I_2| = 1$ ,  $\Omega_{1,2} = \Omega \pm \Delta\Omega$  in dependence on the transverse field  $\Omega$ ;  $\Delta\Omega = 0$  (thin dashed curve),  $\Delta\Omega = 0.5$  (dot-dashed curve),  $\Delta\Omega = 0.9$  (dotted curve),  $\Delta\Omega = 1.1$  (dashed curve),  $\Delta\Omega = 2$  (solid curve). The on-site magnetizations were obtained from the correlation functions  $\langle s_{100}^x s_{600}^x \rangle$  and  $\langle s_{100}^x s_{601}^x \rangle$  computed for chains which consist of  $N = 700$  sites.

magnetization at sites at which the transverse field equals to  $\Omega + \Delta\Omega$  as a function of  $\Omega$  for different values of  $\Delta\Omega = 0, 0.5, 0.9$  ( $\Delta\Omega < |I|$ ) and  $\Delta\Omega = 1.1, 2$  ( $\Delta\Omega > |I|$ ) (for details of numerical calculation see [8]). The data show that for  $\Delta\Omega < |I|$  two phases appear as  $\Omega$  varies: the quantum Ising phase for  $|\Omega| < \sqrt{\Delta\Omega^2 + I^2}$  and the quantum paramagnetic phase otherwise (thin dashed, dot-dashed and dotted curves in Fig. 1). Contrary, for  $\Delta\Omega > |I|$  three phases appear as  $\Omega$  varies: the low-field quantum paramagnetic phase for  $|\Omega| < \sqrt{\Delta\Omega^2 - I^2}$ , the quantum Ising phase for  $\sqrt{\Delta\Omega^2 - I^2} < |\Omega| < \sqrt{\Delta\Omega^2 + I^2}$  and the strong-field quantum paramagnetic phase otherwise (dashed and solid curves in Fig. 1). The quantum phase transitions between different phases are accompanied by divergence of the correlation length  $\xi^x$  and logarithmic singularity of the static trans-

verse susceptibility  $\chi^z$  (these results follow from the numerical data and the analytical formula for the ground-state energy obtained with the help of continued fractions, respectively [6]). Thus, the critical behaviour remains as for the uniform chain. Next we turn to the low-temperature behaviour of the specific heat  $c$ . The existence of zero-energy excitations produces the linear dependence  $c$  versus  $T$  as  $T \rightarrow 0$  which may serve as an indication of the quantum critical point. The exact analytical results for the temperature dependence  $c$  versus  $T$  obtained with the help of continued fractions confirm that there are either two (if  $\Delta\Omega < |I|$ ) or four (if  $\Delta\Omega > |I|$ ) values of transverse field at which  $c$  decays linearly as  $T \rightarrow 0$  [6]. If  $\Delta\Omega = |I|$  besides the critical fields  $\pm\sqrt{2}\Delta\Omega$  one more critical field,  $\Omega_c = 0$ , appears. While  $\Omega$  approaches  $\Omega_c = 0$  the energy gap vanishes  $\sim |\Omega - \Omega_c|^2$  (but not  $\sim |\Omega - \Omega_c|$  as for the discussed before critical points  $\Omega_c$ ) that results in a finite value of  $\chi^z \sim |\Omega - \Omega_c|^2 \ln |\Omega - \Omega_c|$  and a logarithmic singularity of its second derivative at  $\Omega = \Omega_c = 0$ .

Extending the reported analysis for chains of period 3 we find that either two, or four, or six second-order quantum phase transition points may occur. Again the number of the quantum phase transitions is controlled by the strength of nonuniformity and the critical behaviour remains the same as for the uniform case. Moreover, weaker singularities may appear. It is worth to note that the condition (2) may be tuned by arbitrary parameter(s) influencing on-site transverse fields and intersite exchange interactions. For example, for a chain with  $I_{1,2} = I \pm \Delta I$ ,  $\Delta I \geq 0$ ,  $\Omega_n = \Omega$  the change of  $I$  may yield a different number of quantum phase transition points depending on a relation between  $\Delta I$  and  $\Omega$ . Finally, let us mention recent papers [10, 11] where similar questions have been addressed for spin- $\frac{1}{2}$  anisotropic  $XY$  chains.

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